PARAMETRIC STOCHASTIC INSTABILITY OF CONNECTED OSCILLATOR

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Oscillators are used for investigation different physical processes, for example, oscillation in plasma, modeling of continuous medium ets. The large interest is fluctuation influence on oscillation processes. It is well known that multiplicative fluctuation causes the instability in thus systems. We investigated the systems of two and three connected oscillators. We have shown that it is possible considerably to increase the increment of this instability. The conditions when this is possible are obtained. The numerical simulation of processes in the system of two and three connected oscillators was carried out. Oscillation increment was increasing when correspond conditions were performed.

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1. INTRODUCTION

Oscillator is model, which widely is used for studying of different oscillation processes. For example, many of oscillation in plasma physics may be considered as connected oscillators. This may perform by separating temporal and spacing parts in solution of according physical problem. Space part is described by differential equation in partial derivatives. This allows to define space structure of electromagnetic fields. On other hand temporal processes dynamic is described by means temporal ordinary differential equations, which may be considered as describing the system of connected oscillators. Such consideration is possible in complicated electrodynamical structures filled by plasma. For example, cavity filled by plasma. Such system may be considered as two connected oscillators. One of them is electromagnetic, i.e. this is natural oscillation of cavity, and other is oscillation of plasma. This system was considered. It was supposed that plasma density was subjected to fluctuation action.

The dynamic as linear as nonlinear oscillators has enough studied. The different physical systems are subjected action of fluctuations. In oscillation ones they may be as additive as multiplicative. One of the important results is that multiplicative fluctuations may cause instability. It increment usually is proportional to fluctuation level. In the case of multiplicative fluctuations the frequencies one or some oscillators are random functions. This causes instability named as parametric instability one. Ref. [1,2,3] are devoted different physical processes that may be considered as connected oscillators.

But it may be expected increment increasing of such instability in some cases. We investigated some of these ones. We showed that fluctuations instability increment might be proportional to root of some power of fluctuation level. It may be realized at some values of parameters, which describe concrete physical system. Values of these parameters are defined.

First of all we considered two connected oscillators with multiplicative fluctuations. On other hand, we studied three connected oscillators, where each of them is connected with neighboring. Such system may be employed for modeling of continuos matter. We also made some conclusion when increment increase may be realized. Besides, it will be shown that in the system of three connected oscillators increment of stochastic parametric instability may be proportional to root of sixth power of fluctuation level. In the case of two oscillators increment may be proportional to cubic root. Essential increasing of increment was confirmed by numerical simulation.

2. THE SYSTEM OF TWO CONNECTED LINEAR OSCILLATORS

Let us consider two connected linear oscillators, frequency one of them is subjected of random perturbation action. The differential equations describing this system are following:

$$\begin{aligned} &\ddot{\xi}_1 + \xi_1 = -\mu\Omega^2\xi_2 - \alpha\mu\Delta\xi_2, \\ &\ddot{\xi}_2 + \left(\Omega^2 + \Delta\right)\xi_2 = -\mu\xi_1, \end{aligned} \qquad (1)$$

where ξ_1, ξ_2 – generalized coordinates, describing these oscillators, μ – connection coefficient ($0 \le \mu \le 1$), Ω – dimensionless frequency of the second oscillator (the dimensionless time is normalized on the first oscillator period, $\Delta << \Omega^2$ – random addition to second oscillator frequency, α – coefficient accounting additive influence of second oscillator on first one ($0 \le \alpha \le 1$). Random addition is Gaussian and delta-correlated,

$$\langle\Delta(\tau)\Delta(\tau')\rangle = N\delta(\tau - \tau') \qquad (2)$$

where N – noise level.

The differential equation set for first moments which are average values of $\xi_1, \dot{\xi}_1, \xi_2, \dot{\xi}_2$, coincides with set (1) when Δ =0, therefore they are not interest for us. We investigated second moments, which are average values of all products above mentioned $\xi_1, \dot{\xi}_1, \xi_2, \dot{\xi}_2$. The amount of second moments is ten. The differential equations set for them was obtained. The technique of variational derivatives and Furutsu-Novikov formula was used for obtaining of second moments equations set. This technique is described in [2]. Averaging (1) for obtaining equation set for first moments we have addendum $< \xi_2\Delta >$. The set of equations for second moments contains addendum $< \xi_i\xi_k\Delta >$. Thus new unknowns appear. For reducing this expressions following Furutsu-Novikov formula is employed:

$$< \Delta(\tau)\xi_2(\tau) >= \frac{1}{2} N \left\langle \frac{\delta\xi_2(\tau)}{\delta\Delta(\tau)} \right\rangle,$$

where $\left\langle \frac{\delta\xi_2(\tau)}{\delta\Delta(\tau)} \right\rangle$ is variational derivative of $\xi_2(\tau)$ with respect $\Delta(\tau)$ at time τ. This formula is correct for random Gaussian delta-correlated processes. To obtain variational derivative equations (1) are integrated formally.

For solution defining of these equations for second moment in exponential form as $\exp(\lambda\tau)$, the characteristic equation was obtained. It is algebraic equation of tenth power. It may be presented such as consisting of two parts:

$$\prod_{i=1}^{10} (\lambda - \lambda^{(2)}) = NP(\lambda),$$

where $\lambda^{(2)}$ – roots of left part, *N* – fluctuation level. Left part is polynomial. Right part is polynomial too proportional to fluctuation level. The right part of characteristic equation in such form is conditioned by fluctuations influence. The roots of left part polynomial simply are connected with natural frequencies of initial linear connected oscillators system without fluctuations, which describing by set (1) when Δ=0. These are all different sums and differences of natural frequencies multiplied on imaginary one. The characteristic equation for (1) when *N=0* is following:

$$\lambda^4 + \left(1+\Omega^2\right)\lambda^2 + \Omega^2\left(1-\mu^2\right) = 0,$$

and its solutions are:

$$\lambda_{1,2}^{(0)} = \pm \frac{i}{\sqrt{2}} \sqrt{1+\Omega^2 - \sqrt{\left(1-\Omega^2\right)^2 + 4\mu^2\Omega^2}},$$

$$\lambda_{3,4}^{(0)} = \pm \frac{i}{\sqrt{2}} \sqrt{1+\Omega^2 + \sqrt{\left(1-\Omega^2\right)^2 + 4\mu^2\Omega^2}}.$$

They coincide with natural frequencies without imaginary one *i*. Accordingly for $\lambda^{(2)}$ we obtain following expressions:

$$\lambda_{1,2}^{(2)} = \pm\sqrt{2}i\sqrt{1+\Omega^2 - \sqrt{\left(1-\Omega^2\right)^2 + 4\mu^2\Omega^2}}$$

$$\lambda_{3,4}^{(2)} = \pm\sqrt{2}i\sqrt{1+\Omega^2 + \sqrt{\left(1-\Omega^2\right)^2 + 4\mu^2\Omega^2}}$$

$$\lambda_{5,6}^{(2)} = 0$$

$$\lambda_{7,8}^{(2)} = \pm\frac{i}{\sqrt{2}}\left\{\sqrt{1+\Omega^2 - \sqrt{\left(1-\Omega^2\right)^2 + 4\mu^2\Omega^2}} + \sqrt{1+\Omega^2 + \sqrt{\left(1-\Omega^2\right)^2 + 4\mu^2\Omega^2}}\right\}$$

$$\lambda_{9,10}^{(2)} = \pm\frac{i}{\sqrt{2}}\left\{\sqrt{1+\Omega^2 - \sqrt{\left(1-\Omega^2\right)^2 + 4\mu^2\Omega^2}} - \sqrt{1+\Omega^2 + \sqrt{\left(1-\Omega^2\right)^2 + 4\mu^2\Omega^2}}\right\}$$

When noise is low, the roots of characteristic equation may be presented in following form: $\lambda = \lambda^{(2)} + \delta$. The addend δ is conditioned by fluctuation influence. λ is complex value. If real part of λ (i.e. real part δ) is positive, there is instability on the second moments in the system consisting of two connected linear oscillators. This instability is conditioned by fluctuations, and it increment equals real part λ. In general case this increment is proportional to noise level. But it is possible that δ will be proportional to root of some power of fluctuations level. It may be when there are multiple roots among ones of left part polynomial. In this case the addend δ is proportional to root of power, which coincide with the left part equal roots number. In general case the equal roots of left part of characteristic equation may exist at some parameters of initial set (1). To get parameter value when multiple roots exist it is necessary to equate two or more roots $\lambda^{(2)}$. The obtained equalities are equations for parameters defining when equal roots exist. For example, two times multiple roots exist when following condition for μ is right in the case $\alpha = 1$:

$$\mu^2 = \frac{(9\Omega^2 - 1)(9 - \Omega^2)}{100\Omega^2}. \quad (3)$$

Numerical simulation results for this case will be considered in section 5. But it is possible that one or more roots of the right part polynomial $P(\lambda)$ are same as $\lambda^{(2)}$. In this case there is not expected increasing of increment. The values of these parameters were defined. When left part of characteristic equation has multiple roots, increment of mentioned instability becomes more large comparatively the ordinary case. We defined increments of second moment instability, which were proportional as cubic root as square root of fluctuation level. The parameters values, when this is possible, were defined.

The above considered results may be using for concrete physical system. It is cylindrical cavity filled by plasma with fluctuating density. Such system may be considered as two connected linear oscillators. One of them corresponds to electromagnetic oscillations of cavity other corresponds to plasma oscillations. The last is objected to plasma density fluctuations. There is instability on second moments in such system. But there is not expected increase of increment in cavity with fluctuating plasma.

3. SOME CONCLUSION ABOUT ARBITRARY OSCILLATORS NUMBER

Let us consider system of arbitrary number linear connected oscillators described by following differential equations set:

$$\ddot{x}_i + \omega_i^2 x_i = \sum_{j=1, j\neq i} \alpha_{ij} x_j \quad i = 1,2,\ldots,n \quad (4)$$

where x_i – generalized coordinate describing i-th oscillator, ω_i – it's frequency, a_{ij} – connection coefficients, *n* – oscillators number. We suppose that there is frequency fluctuation in one of oscillators. The above mentioned technique of variational derivatives may be used for moments equations set obtaining. It may be shown that first moments equation set coincides with set (4) without fluctuations. The second moments are interesting. The first moments number is $M_1 = 2n$.

It may be shown that second moments number equals $M_2 = n(2n+1)$. The differential equations set may be obtained. It will be linear homogeneous set. Solutions may be presented in exponential form as for two oscillators. There is characteristic equation for second moments. It is algebraic equations of M_2 power. It may be presented in form of two parts, as that performed for two oscillators. The left part of this equation may have more larger number of multiple roots than in two oscillators case. The maximum multiplicity is defined by as oscillators number as parameters number describing this physical system. Maximum multiplicity may equal $k = \min(M_2, M_p)$, where M_p – number of parameters of considered system.

In general case the system containing large number of connected linear oscillators and described by large number of parameters, and subjected action of multiplicative fluctuations may have large multiple roots number of characteristic equation left part for second moments. Solutions of characteristic equation in the case of low fluctuation level may be presented in form $\lambda = \lambda^{(2)} + \delta$, as it was mentioned above. In the case of left part roots with large multiplicity the addend δ will be proportional to root of multiplicity power of fluctuations level. The addend δ may be larger, if multiplicity of corresponding root is large, in the case of many oscillators comparatively case of two oscillators. It was mentioned that the roots λ are complex values. Therefore δ is complex too. If real part δ is positive there is instability on second moments in system of many oscillators with multiplicative fluctuations. If multiplicity of mentioned root is large the instability increment, which equals real part of δ, will be large too. In the case of many oscillators it may be more larger comparatively case of two connected linear oscillators.

4. THREE CONNECTED OSCILLATORS UNDER ACTION OF MULTIPLICATIVE FLUCTUATIONS

In this section we shell be shown that in system of three connected oscillators with multiplicative fluctuations the essential increase of stochastic parametric instability is possible. Such system is describing by following:

$$\ddot{x}_1 + \omega_1^2 x_1 = \alpha_{12} x_2 \,,$$
$$\ddot{x}_2 + \omega_2^2 x_2 = \alpha_{21} x_1 + \alpha_{23} x_3 \,, \qquad (5)$$
$$\ddot{x}_3 + \omega_3^2 x_3 = \alpha_{32} x_2 \,,$$

where symbols are as in (4). The characteristic numbers of this system are when frequencies equal:

$$\lambda_{1,2} = \pm i, \ \lambda_{3,4} = \pm i\sqrt{1+A}, \ \lambda_{5,,6} = \pm i\sqrt{1-A},$$

where $A = a_{12}a_{21} + a_{23}a_{32}$.

We suppose that frequency of first oscillators ω_1 is under action of fluctuations, which are delta-correlative with level N_1. The number of second moments is 21. The differential equations set for them was obtained. The technique of variational derivatives and Furutsu-Novikov formula as in section 2 was used for obtaining of second moments equations set too. As in section 2, the characteristic equation was obtained. It was presented in form:

$$\prod_{i=1}^{21} (\lambda - \lambda^{(2)}) = N_1 P(\lambda) \,. \qquad (6)$$

$\lambda^{(2)}$ are all possible sums of λ_i. The roots were analyzed. The goal of this analysis was to obtain the existence multiple roots among $\lambda^{(2)}$. There are roots with different multiplicity. The parameters values when this is possible were obtained. $\lambda^{(2)}=0$ with maximal multiplicity that equals 9 and $\lambda^{(2)} = \pm 2i$ with multiplicity 6. It possible when parameter *A=0*. But such $\lambda^{(2)}$ may be roots of polynomial $P(\lambda)$. To exhibit this the right part of (6) was transformed. This right part is determinant. The determinant order was reduced to order of three. It was exhibited that polynomial $P(\lambda)$ has multiple roots coinciding with $\lambda^{(2)}$. Just it is 0. As mentioned above in section 2 the addend δ is proportional to root of multiplicity power of noise level for multiple $\lambda^{(2)}$. But this condition is violated when polynomial $P(\lambda)$ has same roots. In this case the root index in the expression for δ is less on number that equals multiplicity of corresponding root of $P(\lambda)$. In our case such root is $\lambda^{(2)}=0$. $\lambda=0$ is fourth degenerated root of polynomial $P(\lambda)$. Therefore $\delta \approx \sqrt[5]{N_1}$ for $\lambda^{(2)}=0$ and $\delta \approx \sqrt[6]{N_1}$ for $\lambda^{(2)} = \pm 2i$. There are complex δ with positive real part. There is instability in this case. Fluctuations are faint. Therefore we have fluctuation increment increasing when multiple roots $\lambda^{(2)}$ exist. Thus as it see from this section there is increasing possibility of stochastic parametric instability increment for three connected oscillators comparatively of two ones case. This confirms conclusions made in section 3. Results of this section may be used when fluctuation influence on continuos matter is investigated. Continuos matter may be presented as chain of connected oscillators.

5. NUMERICAL SIMULATION

In previous sections we analytically investigated some examples when increment of parametric stochastic instability may increase. These results relate to second moments. In [2,3] the stochastic parametric instability for one oscillator was considered in detail. The characteristics of individual realizations were investigated. It was shown that oscillation amplitude in individual realization increases with some increment.

We numerically investigated some cases when essential increasing of increment was expected. We considered two and three connected oscillators for sets (1) and (5). The parameter values corresponded to conditions when instability increment increasing was expected. The corresponding plots are demonstrating our conclusions in section 3.

The random function was taken Gaussian with zero mean value. Our consideration is correct when random function is delta correlated. For numerical simulation this means that correlation radius should be more less then processes period. The correlation radius is of order the mean period of random function. In our cases this period was of order 0.16 whereas processes period was ≈ 2.0.

For two oscillators was considered case for two multiple roots. The corresponding condition given by expression *A=0*. As it was mentioned in section 4 in the case of three oscillators the degree of intrinsic was six. The maximum increment corresponds to the optimal condition (3) for two oscillators and $A=0$ for three ones. Corresponding plots are shown on the fig. 1 for two oscillators and fig. 3 for three oscillators. In these plots there is a typical shape of generalized coordinates dynamic for two and three oscillators.

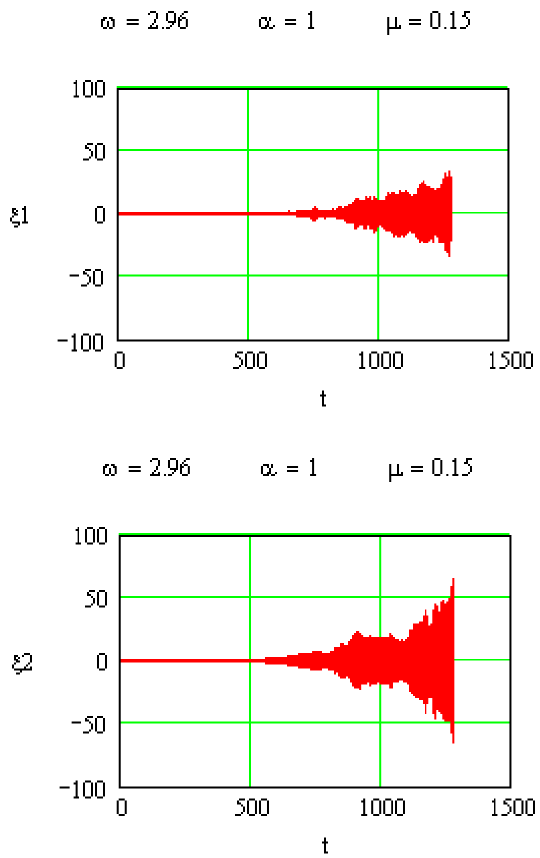

Fig. 1. Dynamic of two connected oscillators in the case of optimal conditions

As it's seen amplitudes of each variables increase with time in averaging. The connection coefficients for three oscillators were following $a_{12} = a_{21} = a_{32} = 0.05$ and $a_{23} = -0.05$.

For others parameter values increments were less. It demonstrated in the fig. 2 for two oscillators and fig. 4 for three oscillators where $a_{12} = a_{21} = a_{32} = 0.05$ and $a_{23} = -0.045$.

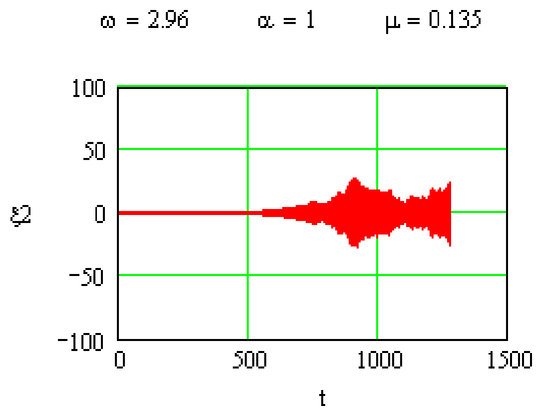

Fig. 2. Dynamic of two oscillators (variable ξ_2) when parameter μ does not correspond to optimal

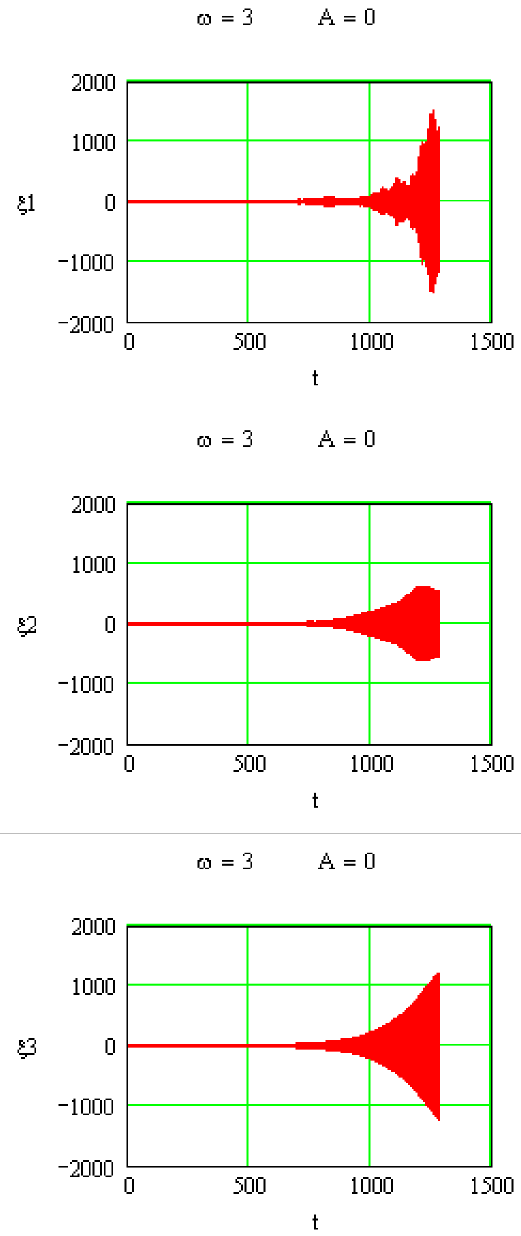

Fig.3. Dynamic of tree connected oscillators in the case of optimal conditions

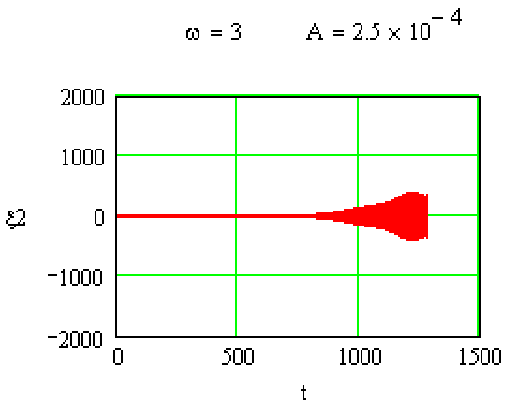

Fig. 4. Dynamic of three oscillators (variable ξ_2) when parameter A does not correspond to optimal

In equations (1) and (5) the nonlinear member was added. In this case the oscillation amplitude is limited.

6. SUMMARY

It is well known that muliplikative fluctuations may cause instability that named parametric stochastic one.. It was shown on the example of one oscillator, which frequency is subjected to random action, in [2]. This instability exists on second moments for delta-correlation fluctuation. In this repot possibility of stochastic parametric instability increment increasing was considered. It has been noted that such increasing may take place in a system of large oscillators number, which described by large parameters number. It was considered on the examples of two and three connected oscillators with multiplicative fluctuations. It is necessary to note that in the case of three oscillators instability increments may be more larger. The necessary conditions of increment increasing were investigated. The parameters when increment increasing takes place were defined. The corresponding numerical simulation was performed. It confirms the analytical conclusion

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